Discrete Uniformization
and
Ideal Hyperbolic Polyhedra

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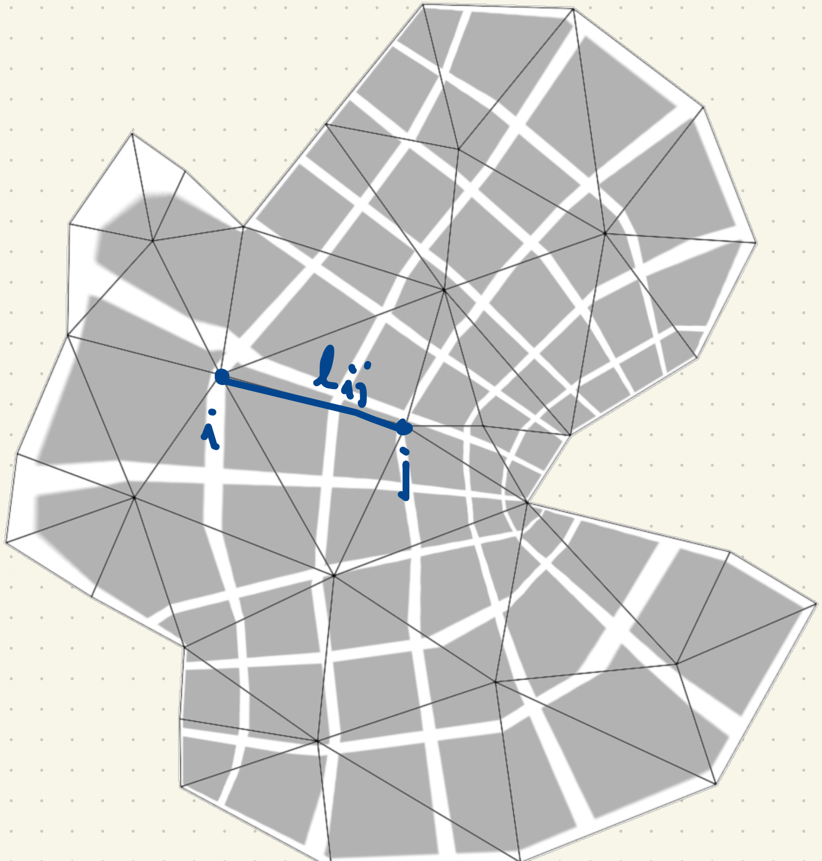
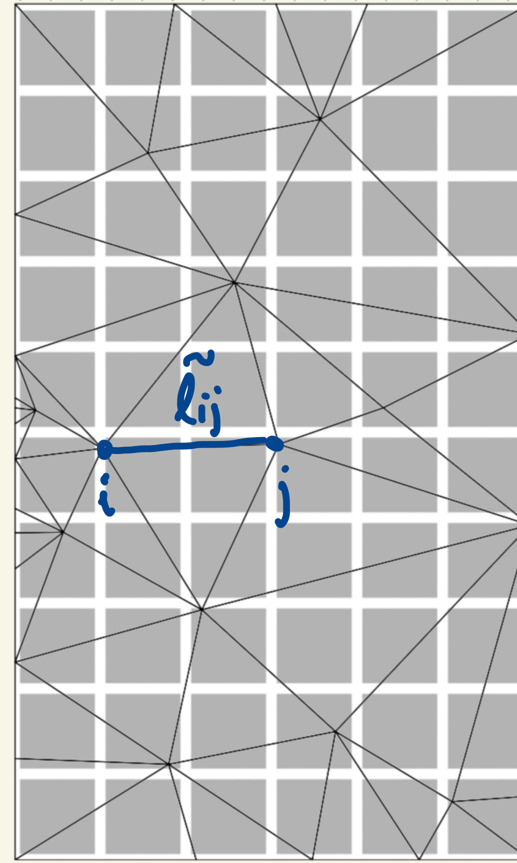
Circle packing and geometric rigidity, ICERM, Jul 6-10 2020

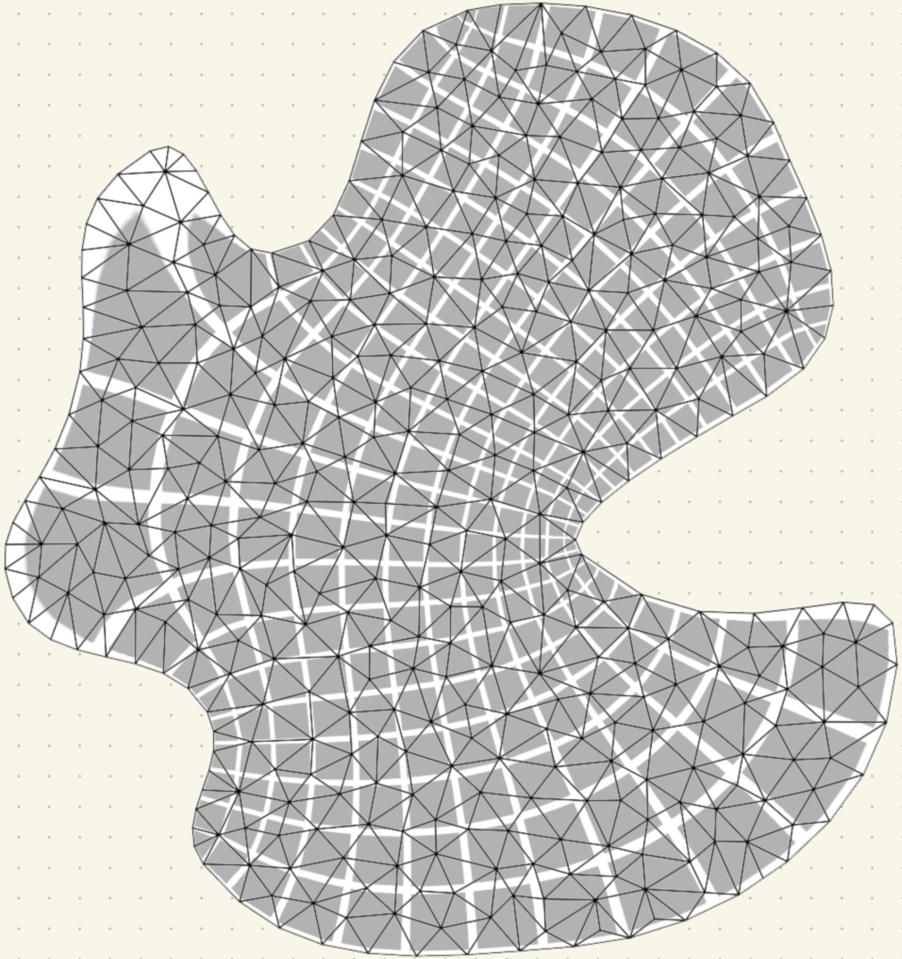
1.

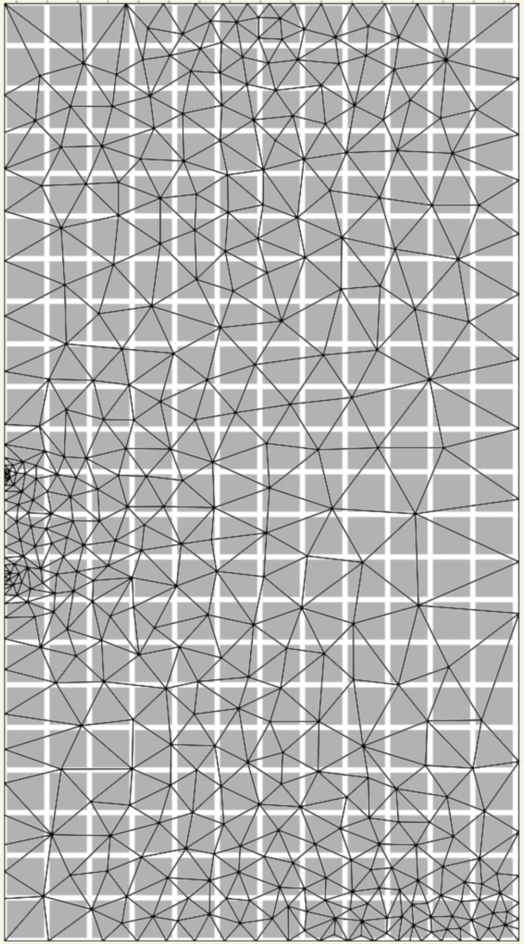
Discrete conformal maps and discrete uniformization.

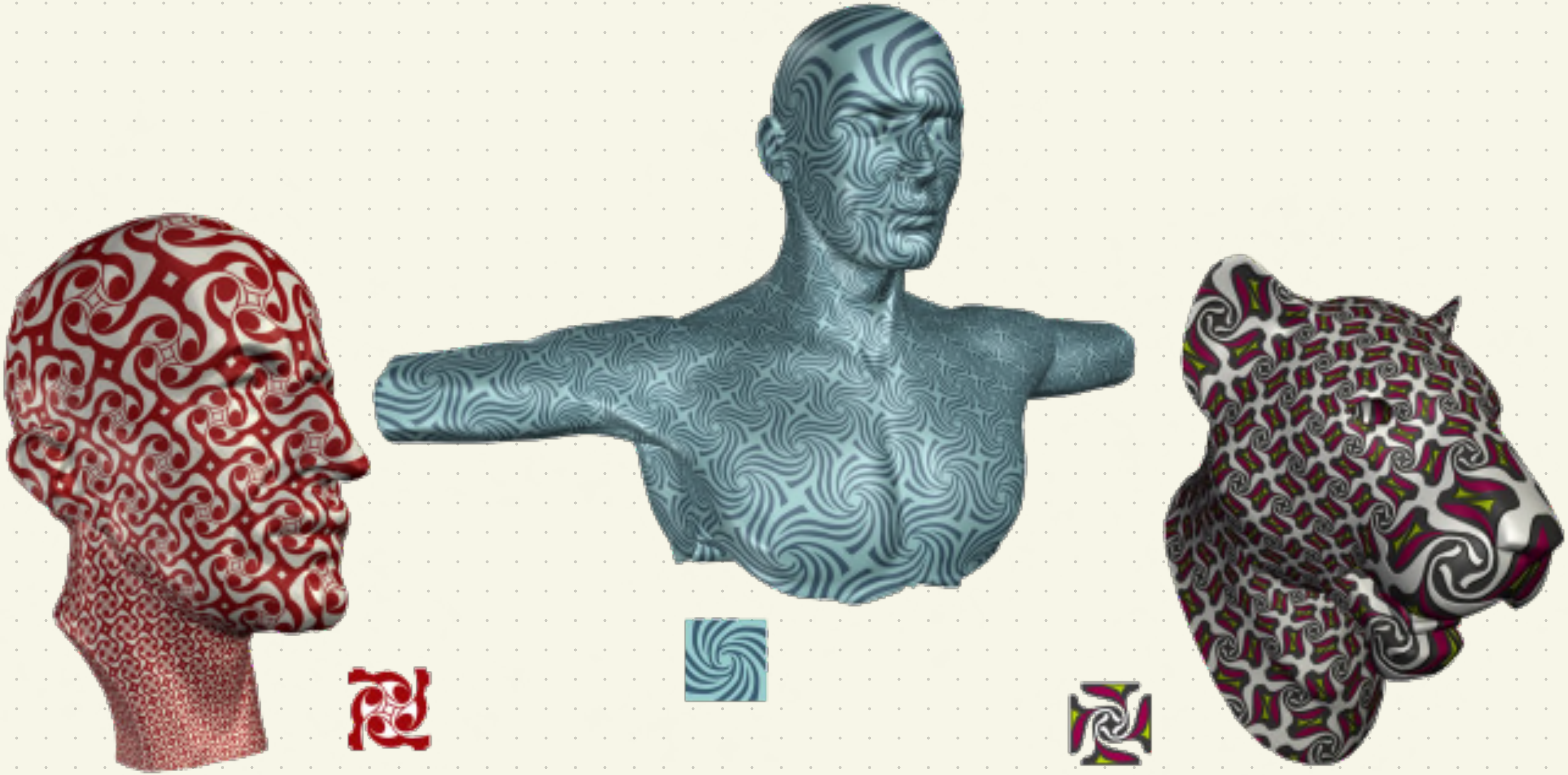




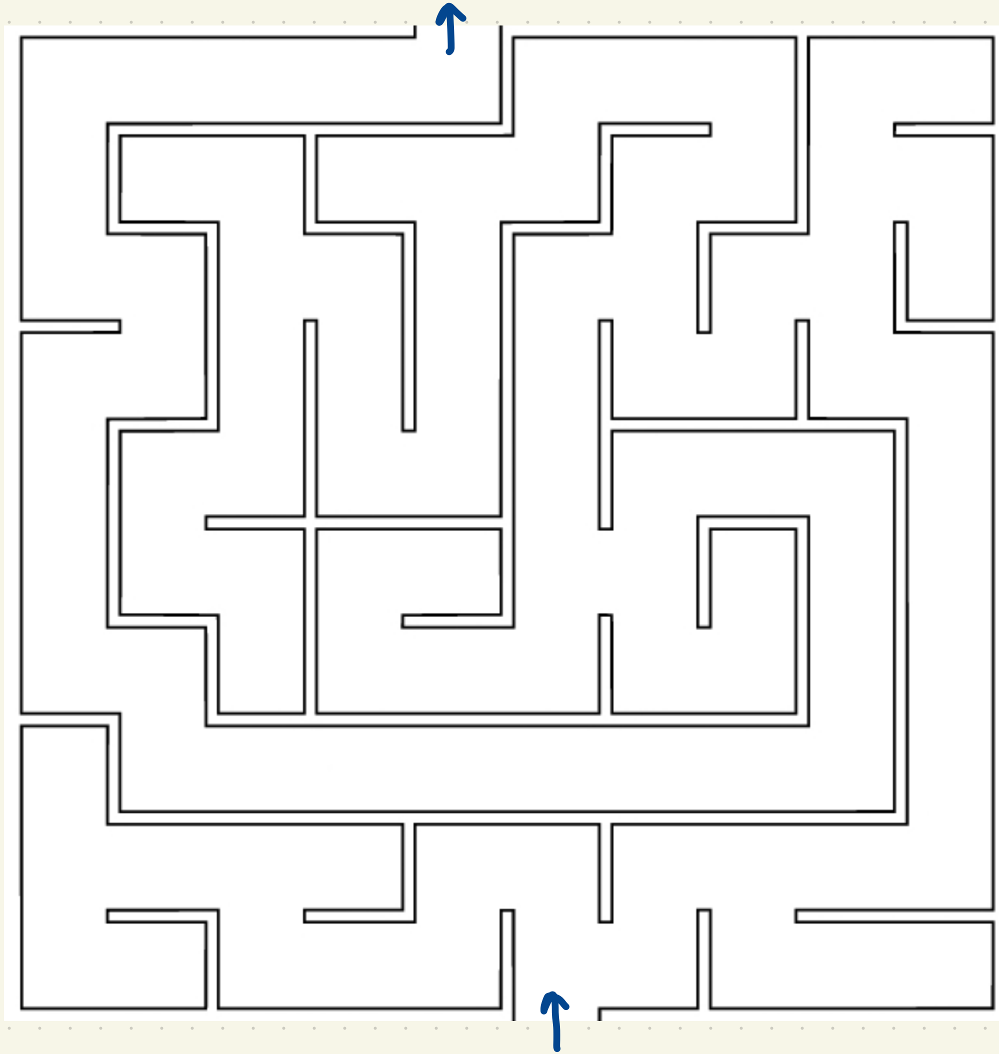
$$\tilde{l}_{ij} = e^{\frac{1}{2}(u_i+u_j)}\, l_{ij} = e^{\frac{1}{2}u_i} \cdot e^{\frac{1}{2}u_j}\, l_{ij} = \sqrt{e^{u_i} \cdot e^{u_j}}\, l_{ij} \quad \text{(Luo)}$$

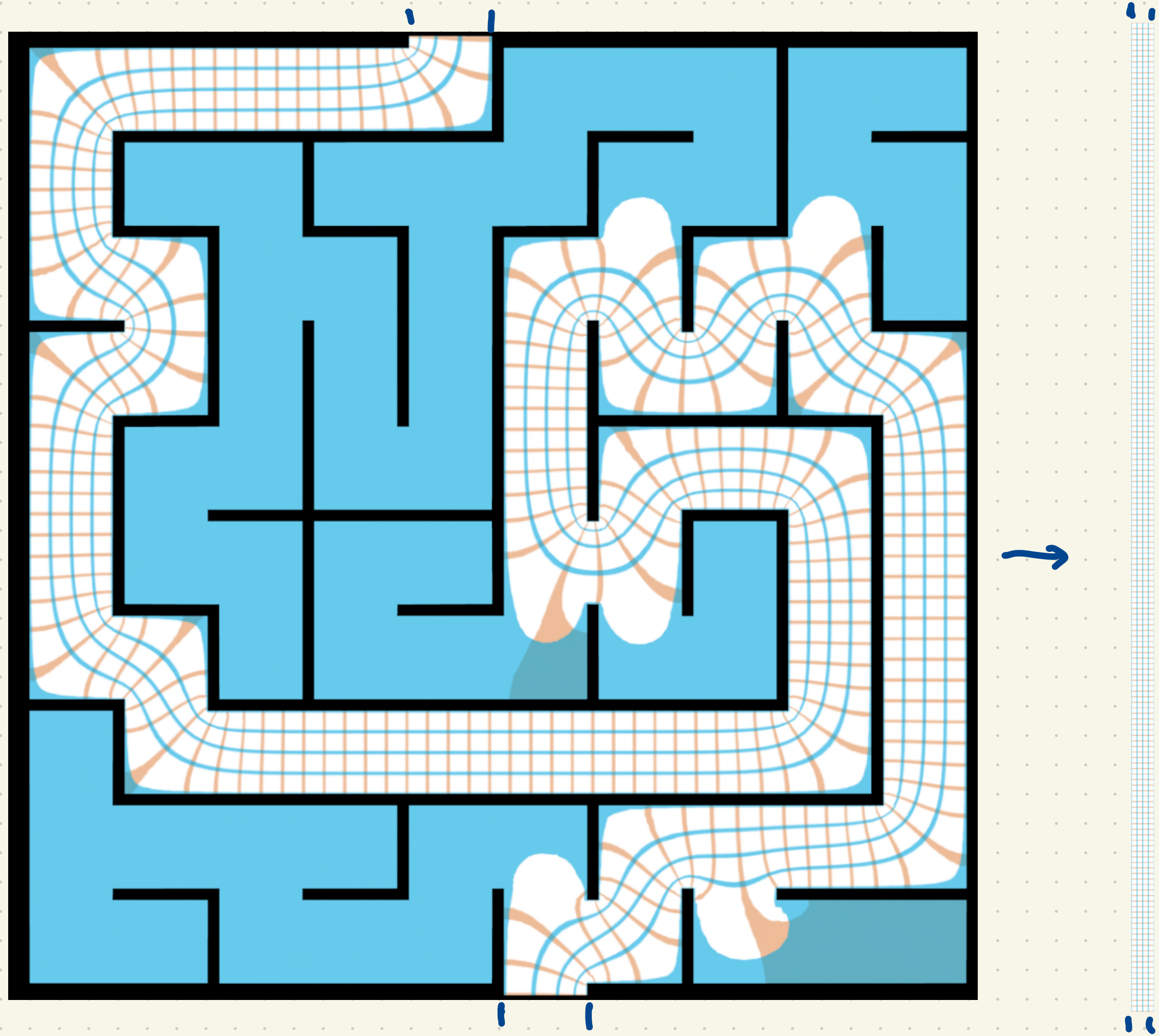






[Springborn, Schröder, Pinkall. Conformal Equivalence of Triangle Meshes]





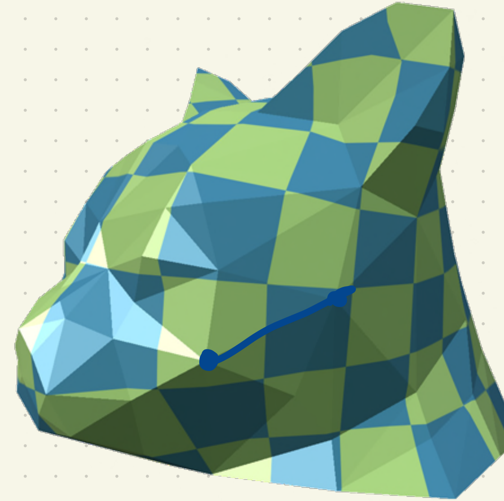
Discrete conformal equivalence v.1

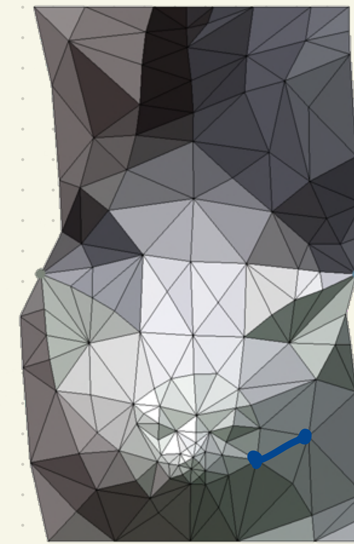
Two combinatorially equivalent triangle meshes (T, ℓ), $(T, \tilde{\ell})$ are <u>discretely conformally equivalent</u> if there is a function

$$u : V \longrightarrow \mathbb{R}$$

such that

$$\tilde{\ell}_{ij} = e^{\frac{1}{2}(u_i + u_j)} \, \ell_{ij}$$

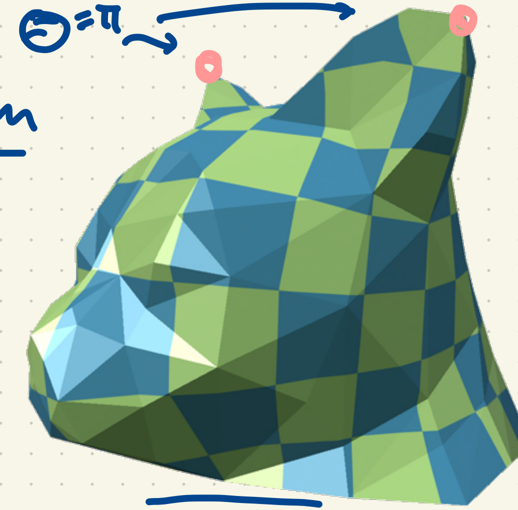




T - abstract triangulation.

$\ell : E \to \mathbb{R}$ edge length function.

Discrete conformal mapping problem



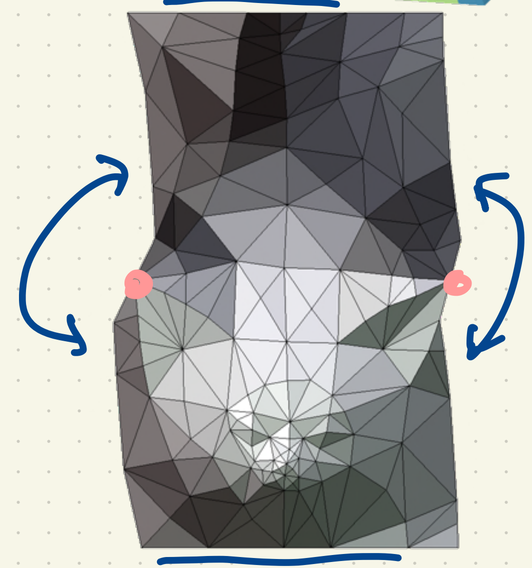
Given
- a triangle mesh (T, ℓ)
- desired angle sums

  $\Theta : V \to \mathbb{R}_{>0}$

  (satisfying Gauss-Bonnet-Condition)

Find a discr. conf. eq. mesh $(T, \tilde{\ell})$ with the given angle sums Θ.

This means: Find $u : V \to \mathbb{R}$.



Variational principle

$$E_{T,\ell,\Theta} : \mathbb{R}^V \longrightarrow \mathbb{R}$$
$$u \longmapsto E_{T,\ell,\Theta}(u)$$

Solutions of mapping problem $\Rightarrow$ $\operatorname{grad} E_{T,\ell,\Theta}(u) = 0$.

$$\operatorname{grad} E_{T,\ell,\Theta}(u) = 0 \quad \Rightarrow \quad \begin{cases} \tilde{\ell} \text{ satisfies all triangle ineq.} \to \text{solution} \\ \text{triangle inequalities violated} \to \text{no solution exists} \end{cases}$$

Uniqueness: Yes. Existence: No.

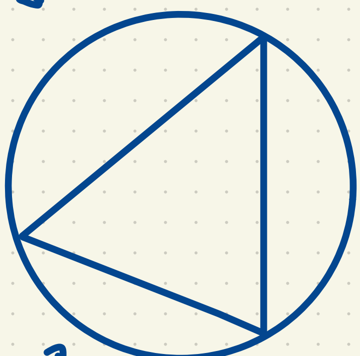
$E_{T,\ell,\Theta}(u)$ is convex !

To solve the discr. conformal mapping problem, minimize $E_{T,\ell,\Theta}(u)$.

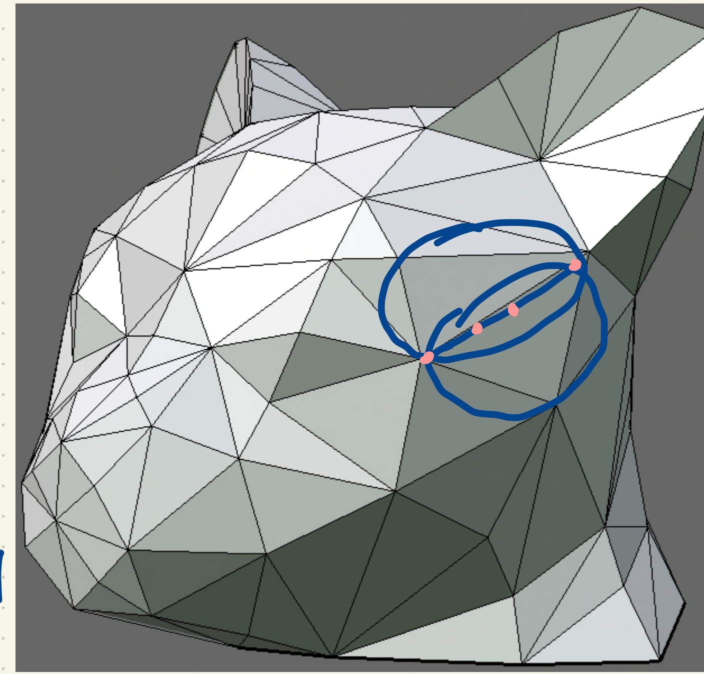
$\rightsquigarrow$ Solution of mapping problem is unique up to scale.
(If it exists.)

The hyperbolic metric induced by circumcircles

euclidean triangle and circumcircle



hyperbolic plane in Beltrami-Klein-model
and an ideal triangle



Discrete conformal equivalence & hyperbolic geometry

Essential observation:

Two combinatorially equivalent triangle meshes are discretely conformally equivalent if and only if they are isometric with respect to the induced hyperbolic metric.

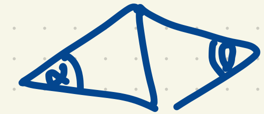
[Bobenko, Pinkall, S]

Discrete conformal equivalence v.2

surface $\qquad$ vcs discrete

Two piecewise flat metrics $d, \tilde{d}$ on (S, V) are discretely conformally equivalent if the hyperbolic metrics induced by the Delaunay triangulations of of (S, V, d) and $(S, V, \tilde{d})$ are isometric.

Equivalently:

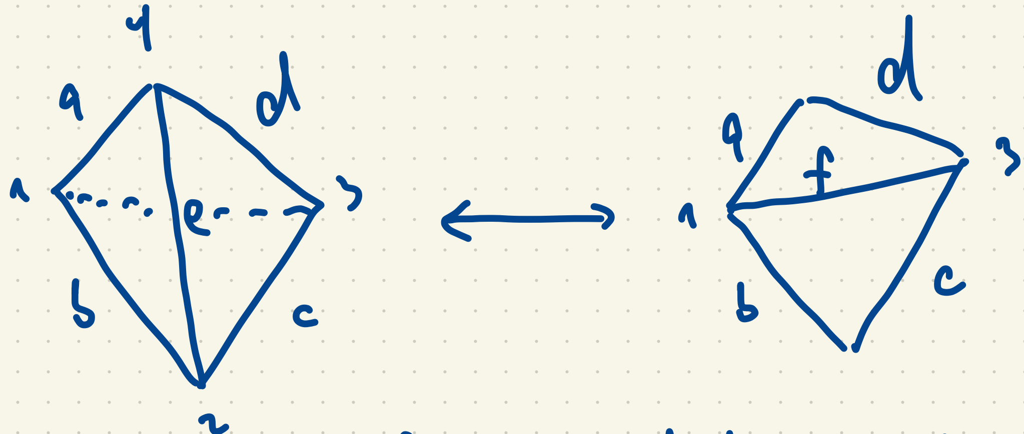


$\alpha + \beta < \pi$

Two Delaunay triangle meshes (T, ℓ), $(\tilde{T}, \tilde{\ell})$ are discretely conformally equivalent if they are related by a combination of vertex scaling,

$$\ell_{ij} \longrightarrow \hat{\ell}_{ij} = e^{\frac{1}{2}(u_i + u_j)} \ell_{ij},$$

and Ptolemy-flips:

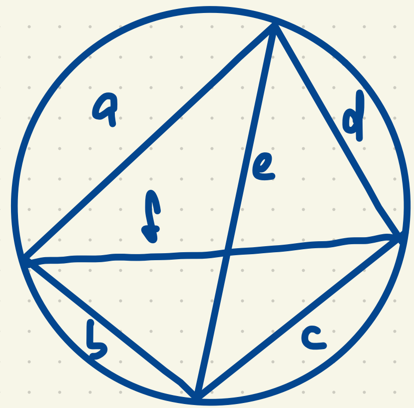


$$ef = ac + bd \qquad \text{(Ptolemy relation)}$$

<u>Equivalent:</u> Definition of Gu, Luo, Sun, Wu.

Ptolemy flips...

- change shapes of triangles unless they share circumcircle



Ptolemy's Theorem:

$$ef = ac + bd$$

- do not change the induced hyperbolic metric
- commute with vertex scaling

Existence & uniqueness!

surface $\quad V \subset S \quad$ metric $\quad$ Gauss-Bonnet

Given: • closed piecewise flat surface (S, V, d)

• desired cone angles $\Theta : V \to \mathbb{R}_{>0}$, $\sum_v (2\pi - \Theta_v) = 2\pi\chi$

There exists a unique piecewise flat metric $\tilde{d}$ on (S, V)

that is: • discretely conformally equivalent and

• has cone angles Θ.

[Gu, Luo, Sun, Wu]

- For $\Theta = 2\pi$, genus $= 1$:

  Discrete Uniformization Theorem for Tori

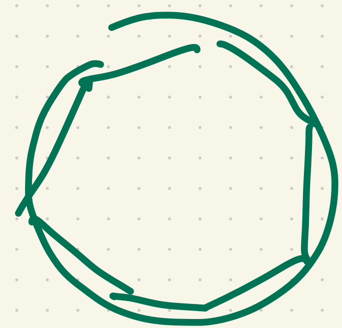
- For higher genus:

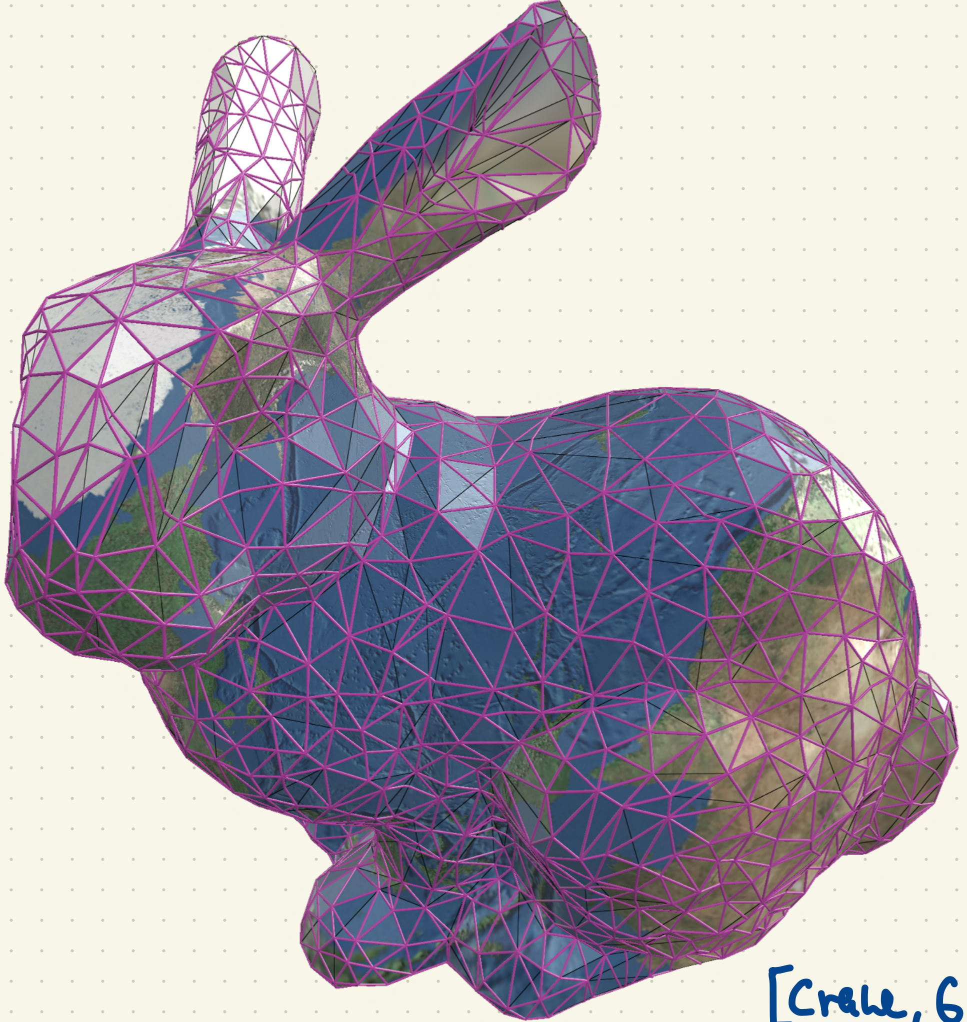
  Analogous theory for piecewise hyperbolic surfaces.

  [Bobenko, Pinkall, S]
  [Gu, Guo, Luo, Sun, Wu] → Uniformization Theorems
  [Prosanov] →

- Genus 0: Discrete Uniformization of Spheres

<u>Theorem</u>. For every piecewise flat metric on (S^2, V), [2-sphere]
there is a convex polyhedron inscribed in S^2
that is discretely conformally equivalent. It is
unique up to a Möbius transformation applied
to the vertices.





[Crane, Gillespie, S] (in prep.)

- Variational principle extends.

2.

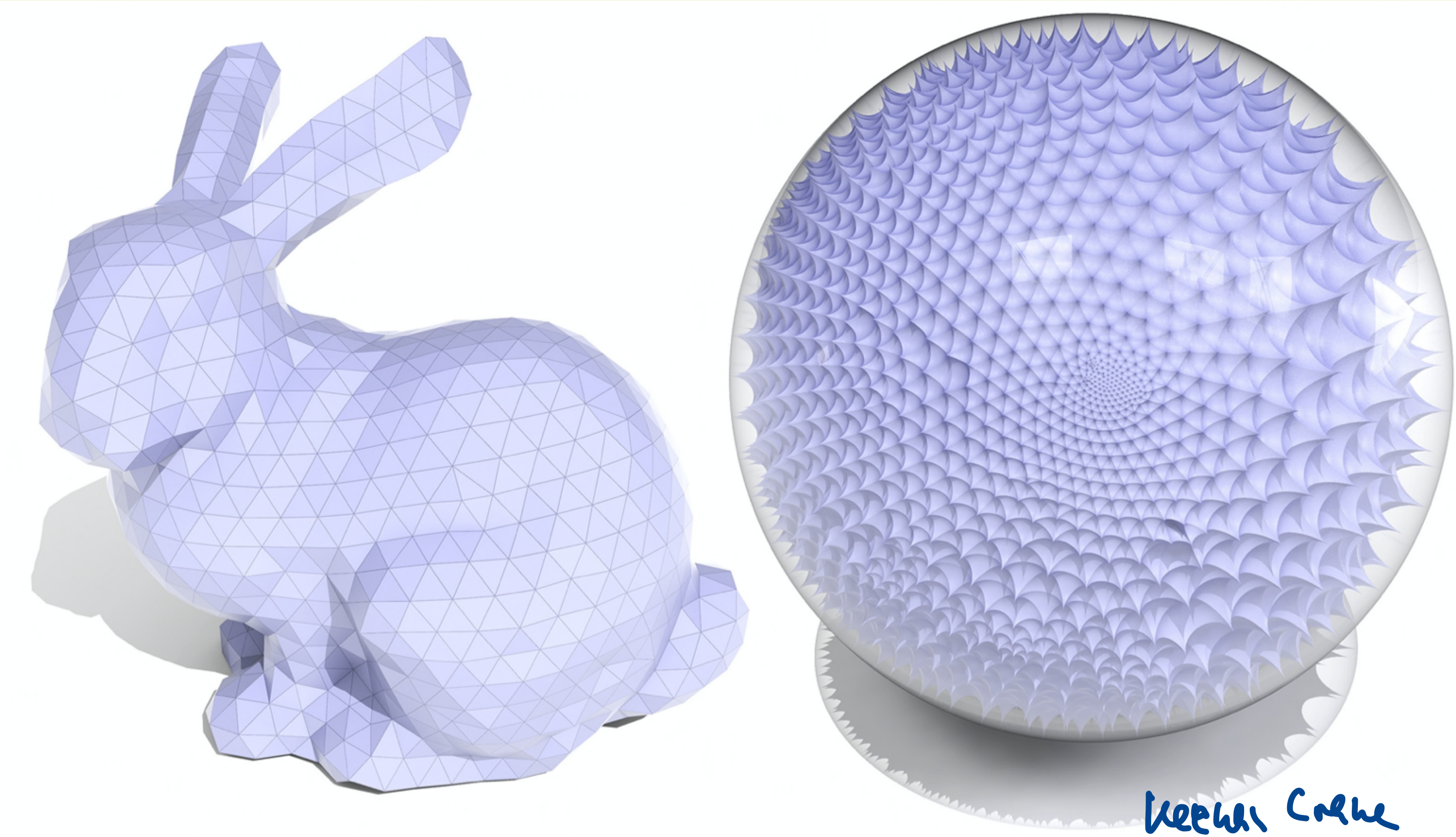
Ideal hyperbolic polyhedra

Discrete Uniformization Theorems

are equivalent to

Realization Theorems for Ideal Hyperbolic Polyhedra

(with prescribed intrinsic metric)


Keenan Crane

<u>Theorem</u> (Rivin). For every complete hyperbolic metric of finite area on $S^2 \setminus V$, there is a unique convex ideal polyhedron in H^3 realising the metric.

For tori & higher genus: Schlenker, Fillastre